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Solutions for periodic Schrödinger equation with spectrum zero and general superlinear nonlinearities

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ABSTRACT

In this paper we consider the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $V(x)$ and $g(x,u)$ are periodic with respect to x and 0 is a boundary point of the spectrum $\sigma(-\Delta + V)$. Replacing the classical Ambrosetti–Rabinowitz superlinear assumption on $g(x,u)$ by a general super-quadratic condition, we are able to obtain the existence of nontrivial solutions.

1. Introduction and main results

We consider the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a potential and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonlinear coupling which is superlinear as $|u| \to \infty$.

Eq. (1.1) appears in several applications from mathematical physics. For instance, when we look for standing wave solutions of the following time dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + W(x)\psi - f\big(x, |\psi|\big)\psi.$$

It is obvious that standing wave $\psi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$ solves the above equation if and only if $u(x)$ solves (1.1) with $V(x) = \frac{2m}{\hbar^2}(W(x) - E)$ and $g(x,u) = \frac{2m}{\hbar^2} f(x, |u|)u$.

As we know, the nonlinear Schrödinger equation with periodic potential and nonlinearities has been widely investigated for both its importance in applications and mathematical interest, see, e.g., [1,4–10,13,14,17,21]. It is well known (see, e.g., [18]) that the spectrum of the self-adjoint operator $A = -\Delta + V$ in $L^2(\mathbb{R}^N)$ is purely continuous and may contain gaps, i.e. open intervals free of spectrum. Here we recall some results on existence and multiplicity of solutions of such an equation depending on the location of 0 in $\sigma(A)$.

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Case 1. $0 < \inf \sigma(A)$. In [6] Coti-Zelati and Rabinowitz proved the existence of infinitely many solutions provided $g \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies suitable growth condition and the Ambrosetti–Rabinowitz condition [2]: there is $\mu > 2$ such that

$$0 < \mu G(x,u) \leqslant g(x,u)u \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R} \setminus \{0\}. \tag{1.2}$$

As we know (1.2) plays an important role in studying variational problems and many efforts have been done to weak condition (1.2). Recently, replacing (1.2) by a general superlinear assumption and monotone condition on $g(x,u)$, Li, Wang and Zeng [10] studied the existence of ground state solutions by concentration compactness arguments. We also refer readers to [3,12] where the Ambrosetti–Rabinowitz condition plays an important role.

Case 2. 0 *lies in a gap of the spectrum* $\sigma(A)$. When (1.2) is satisfied and $G(x,u)$ is strictly convex, existence and multiplicity of solutions of (1.1) were established in Alama and Li [1], Buffoni et al. [5] by virtue of a mountain-pass reduction. Without the convexity, by using a generalized linking argument together with a weaker topology setting, Kryszewski and Szulkin [9] obtained the existence, and multiplicity provided $g(x,u)$ is odd in u, of solutions of (1.1). Letting $E = H^1(\mathbb{R}^N)$, in [13], Pankov introduced the following C^1-Nehari type manifold

$$\mathcal{M} := \big\{u \in E \setminus E^-:\ u \neq 0,\ \big(\Phi'(u), u\big) = 0 \text{ and } \big(\Phi'(u), v\big) = 0 \text{ for all } v \in E^-\big\}$$

by assuming additionally that $g \in C^1$, $|g_u(x,u)| \leqslant a'(1 + |u|^{p-2})$ and

$$0 < g(x,u) \leqslant \theta g_u(x,u)u \quad \text{for some } \theta \in (0,1) \text{ and } u \in \mathbb{R} \setminus \{0\} \tag{1.3}$$

(E^- is the negative space of A). Then he obtained the existence of ground state solution by periodic approximation technique. However, if (1.3) does not hold, we know $\mathcal{M}$ is not a C^1-manifold and Pankov's method does not apply any longer. In a recent paper [16], Szulkin and Weth developed a new approach based on reduction the strongly indefinite problem to a definite one and proved the existence of ground state solution with (1.2) replaced by a general super-quadratic condition and monotone condition on $g(x,u)$. There they also proved the existence of infinitely many geometrically distinct solutions for odd nonlinearity.

Case 3. 0 *is a boundary point of a gap of* $\sigma(A)$. As far as we know, there are only two papers [4,14] dealt with this case. In [4] Bartsch and Ding introduced the following conditions:

(V_1) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $(0, \beta] \cap \sigma(A) = \emptyset$;

(g_1) There are constants $a_1 > 0$ and $2 < \gamma \leqslant \mu < 2^*$ such that
$$a_1|u|^\mu \leqslant \gamma G(x,u) \leqslant g(x,u)u \quad \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R};$$

(g_2) There are constants $a_2 > 0$ and $2 < p \leqslant q < 2^*$ such that
$$\big|g(x,u)\big| \leqslant a_2\big(|u|^{q-1} + |u|^{p-1}\big) \quad \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

With (V_1), (g_1), (g_2), the authors were able to prove the existence of weak solution $u \in H^2_{loc}(\mathbb{R}^N)$ and $u \in L^t(\mathbb{R}^N)$ for $\mu \leqslant t \leqslant 2^*$. There they also showed the existence of infinitely many geometrically distinct solutions for odd nonlinearity. Later, this result was improved by Willem and Zou in [14] by using an improved generalized linking theorem; instead of conditions (g_1), (g_2), the authors obtained the existence of one weak solution by assuming:

(s_1) There are constants $c_1, c_2 > 0$ and $2 < \mu < 2^*$ such that
$$c_1|u|^\mu \leqslant g(x,u)u \leqslant c_2|u|^\mu \quad \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

(s_2) $g(x,u)u - 2G(x,u) > 0$ for all $x \in \mathbb{R}^N$, $u \in \mathbb{R} \setminus \{0\}$.

(s_3) $\liminf_{u \to 0} \frac{g(x,u)u}{G(x,u)} \geqslant \gamma$ uniformly for all $x \in \mathbb{R}^N$.

(s_4) There exists $c > 0$ such that
$$\liminf_{u \to \infty} \frac{g(x,u)u - 2G(x,u)}{|u|^\alpha} \geqslant c$$
uniformly for all $x \in \mathbb{R}^N$, where $\alpha > \mu^* := \frac{2^*\mu(\mu-2)}{2^*\mu - 2^* - \mu}$, if $2^* < \infty$; $\alpha > \frac{\mu(\mu-2)}{\mu-1}$, if $2^* = \infty$.

As we know, the Ambrosetti–Rabinowitz type superlinear condition plays an important role in proving the boundedness of the $(P.S)^*$ sequence or $(P.S)$ sequence in [4,14]. Thus it is natural to ask if the existence results established in [10,16] still hold when 0 is a boundary point of the spectrum $\sigma(A)$.

The purpose of this paper is to study the case when 0 is a boundary point of the spectrum $\sigma(A)$ with the nonlinearities $g(x,u)$ satisfying general superlinear condition and Nehari type monotone condition. The assumptions are the following:

(G_1) $g(x,u)$ is 1-periodic in x_j for $j = 1, \ldots, N$, $|g(x,u)| \leqslant a(1 + |u|^{p-1})$ for some $a > 0$ and $2 < \mu < p \in (2, 2^*)$ where $2^* := \frac{2N}{N-2}$ if $N \geqslant 3$ and $2^* := +\infty$ if $N = 1$ or 2;

(G_2) $g(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in x;

(G_3) $G(x,u) \geqslant c_0|u|^{\mu}$, here $G(x,u)$ is the primitive function of g;
(G_4) $u \to \frac{g(x,u)}{|u|}$ is strictly increasing on $(-\infty,0)$ and $(0,+\infty)$.

The proofs of the main result are based on variational methods applied to the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) - \int_{\mathbb{R}^N} G(x,u). \tag{1.4}$$

The hypotheses on $g(x,u)$ imply that $E \to \mathbb{R}$ is of class C^1 and that critical points of Φ are solutions of (1.1). By assumption (V_1), we have $E = E^- \oplus E^+$ corresponding to the decomposition of $\sigma(A)$ into $\sigma(A) \cap (-\infty,0]$ and $\sigma(A) \cap [\beta,+\infty)$. If we define a new norm $\|\cdot\|_X$ on $E^{\pm}$ by setting

$$\left\|u^{\pm}\right\|_X^2 = \pm\int_{\mathbb{R}^N} \left(\left|\nabla u^{\pm}\right|^2 + V(x)\left|u^{\pm}\right|^2\right), \quad \text{for } u^{\pm} \in E^{\pm},$$

then Φ can be written as

$$\Phi(u) = \frac{1}{2}\left(\left\|u^+\right\|_X^2 - \left\|u^-\right\|_X^2\right) - \Psi(u), \tag{1.5}$$

where $u = u^- + u^+ \in E^- \oplus E^+$, $\Psi(u) = \int_{\mathbb{R}^N} G(x,u)$. However, since 0 is a boundary point of the spectrum $\sigma(A)$, we know norm $\|\cdot\|_X$ is not equivalent to the standard norm on E, thus the generalized linking theorem in [9,20] cannot be directly used. What's more, we cannot look for solutions of (1.1) in the completion X of E under norm $\|\cdot\|_X$, because $\int_{\mathbb{R}^N} G(x,u)$ is not well defined due to our assumptions on $g(x,u)$. It is well known that without condition (1.2), it is not easy to obtain the boundedness of the Palais–Smale sequence when the action functional is strongly indefinite, especially for the case 0 as a boundary point of the spectrum $\sigma(A)$.

Let X_{μ} be the completion of X under norm $\|\cdot\|_{X_{\mu}} = (\|\cdot\|_X^2 + |\cdot|_{\mu}^2)^{\frac{1}{2}}$. The main result of this paper is

Theorem 1.1. *If assumptions (V_1) and (G_1)–(G_4) are satisfied, then problem (1.1) has at least one solution in X_{μ}.*

At the end of this paper, we can also deal with the case 0 as a left end point of $\sigma(A)$.

(V_2) $0 \in \sigma(A)$ and there exists $\beta > 0$ such that $[-\beta,0) \cap \sigma(A) = \emptyset$.

Theorem 1.2. *If assumption (V_2) holds and $-g$ satisfies (G_1)–(G_4), then problem (1.1) has at least one solution in X_{μ}.*

Motivated by the above recent works, we are going to consider the case 0 as a boundary point of the spectrum $\sigma(A)$ and find at least one ground state solution if the nonlinearities $g(x,u)$ satisfy general superlinear condition and Nehari type monotone condition. The main idea here lies in an approximation technique and an application of a variant generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou [17] (see also Willem and Zou [14], Szulkin and Zou [15]). In detail, we first restrict the action functional Φ on suitable subspace X_n of E and find a critical point u_n. Then with this special $(P.S)_c^*$ sequence $\{u_n\}$, we solve the problem by concentration compactness arguments. In [17,14,15], the authors developed monotonicity trick for strongly indefinite problem, the idea of monotone trick was firstly introduced by [19] and later developed by Jeanjean [8] for Landesman–Lazer type problems in $\mathbb{R}^N$.

The paper is organized as follows. In Section 2, we introduce the variational framework and main variational tool. In Section 3, we prove the existence of critical points for the functional restricted on suitable subspace of $E = H^1(\mathbb{R}^N)$. In Section 4, we prove the existence result for Eq. (1.1) with the critical points obtained in Section 3.

2. The variational framework

Throughout this paper we denote by $|\cdot|_q$ the usual L^q-norm and C for generic constants. Under assumption (V_1), $A = -\Delta + V$ is a self-adjoint operator acting on $H = L^2(\mathbb{R}^N,\mathbb{R}) = L^2$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N,\mathbb{R})$. Letting $(P_\lambda : H \to H)_{\lambda\in\mathbb{R}}$ denote the spectral family of A and setting $H^- := P_0H$ and $H^+ := (Id - P_0)H$, we have the orthogonal decomposition

$$H = H^- \oplus H^+, \qquad u = u^- + u^+.$$

Let $X = \mathcal{D}(|A|^{1/2})$ be equipped with the inner product

$$(u,v) = \left(|A|^{1/2}u, |A|^{1/2}v\right)_{L^2}$$

and norm $\|u\|_X = ||A|^{1/2}u|_2$ where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. We have the decomposition

$$X = X^- \oplus X^+ \quad \text{where } X^{\pm} = X \cap H^{\pm},$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Since the spectrum of A restricted to H^+ is contained in $(\beta, +\infty)$, hence the norm $\|\cdot\|_X$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on X^+, so $X^+ = E \cap H^+$. However, on the subspace $E \cap H^-$, the $\|\cdot\|_X$ is weaker than $H^1(\mathbb{R}^N)$ norm and $E \cap H^-$ is not complete with respect to $\|\cdot\|_X$. To solve the problem in $H^1(\mathbb{R}^N)$, we set, for each $n \in N$,

$$X_n^- := X^- \cap P_{-1/n} H \subset X^- \quad \text{and} \quad X_n = X_n^- \oplus X^+ \subset X.$$

Since the spectrum of A restricted X_n is bounded away from 0 we know $\|\cdot\|_X$ is equivalent to the $H^1(\mathbb{R}^N)$ norm on X^-, hence $X_n \subset E \subset X$. Let

$$Q_n := P_{-1/n} + (Id - P_0) : X \to X_n$$

denote the orthogonal projection. Then we have for any $u \in E$:

$$Q_n u \to u, \quad \text{as } n \to \infty, \text{ with respect to } \|\cdot\|_X \text{ and } |\cdot|_{L^t},\ 2 \leqslant t < 2^*.$$

For each n we define the functional $\Phi_n = \Phi|_{X_n}$, $\Psi_n = \Psi|_{X_n}$, then Ψ_n is well defined in X_n, moreover $\Phi_n, \Psi_n \in C^1(X_n, \mathbb{R})$ and

$$\big(\Psi_n'(u), v\big) = \int_{\mathbb{R}^N} g(x,u)v, \qquad \big(\Phi_n'(u), v\big) = (Au, v) - \int_{\mathbb{R}^N} g(x,u)v.$$

The following abstract critical point theorem plays an important role in proving our main result.

Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp$, $N \subset E$ is a closed and separable subspace. There existing norm $|v|_w$ satisfies $|v|_w \leqslant \|v\|$ and induces a topology equivalent to the weak topology of N on bounded subset of N. For $u = v + w \in E = N \oplus N^\perp$ with $v \in N$, $w \in N^\perp$, we define $|u|_w^2 = |v|_w^2 + \|w\|^2$. Particularly, if $(u_n = v_n + w_n)$ is $|\cdot|_w$-bounded and $u_n \xrightarrow{|\cdot|_w} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E (cf. [17]).

Let $E = E^- \oplus E^+$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N := E^- \oplus Rz_0$ and $E_1^+ := N^\perp = (E^- \oplus Rz_0)^\perp$. For $R > 0$, let

$$Q := \{u := u^- + sz_0\colon\ s \in R^+,\ u^- \in E^-,\ \|u\| < R\}$$

with $p_0 = s_0 z_0 \in Q$, $s_0 > 0$. We define

$$A := \partial Q \quad \text{and} \quad B := \{u := sz_0 + w^+\colon\ s \geqslant 0,\ w^+ \in E_1^+,\ \|sz_0 + w^+\| = s_0\}.$$

For $\Phi \in C^1(E, R)$, define

$$\begin{aligned} \Gamma := \Big\{h \;\Big|\; & h : [0,1] \times \bar{Q} \mapsto E \text{ is } |\cdot|_w\text{-continuous},\ h(0,u) = u,\ \Phi\big(h(s,u)\big) \leqslant \Phi(u),\ \forall u \in \bar{Q}. \\ & \text{For any } (s_0, u_0) \in [0,1] \times \bar{Q},\ \text{there is a } |\cdot|_w\text{-neighborhood } U_{(s_0,u_0)}, \\ & \text{such that } \big\{u - h(t,u)\colon\ (t,u) \in U_{(s_0,u_0)} \cap \big([0,1] \times \bar{Q}\big)\big\} \subset E_{fin}\Big\} \end{aligned}$$

where E_{fin} denotes various finite-dimensional subspaces of E, $\Gamma \neq \emptyset$ since $id \in \Gamma$.

The variant weak linking theorem is:

Lemma 2.1. *(See [17].) The family of C^1-functional $\{\Phi_\lambda\}$ has the form*

$$\Phi_\lambda(u) := I(u) - \lambda J(u), \quad \forall \lambda \in [1,2].$$

Assume

(a) $J(u) \geqslant 0$, $\forall u \in E$, $\Phi_1 = \Phi$;
(b) $I(u) \to \infty$ *or* $J(u) \to \infty$ *as* $\|u\| \to \infty$;
(c) Φ_λ *is* $|\cdot|_w$*-upper semicontinuous,* Φ_λ' *is weakly sequentially continuous on* E*. Moreover,* Φ_λ *maps bounded sets to bounded sets*;
(d) $\sup_A \Phi_\lambda < \inf_B \Phi_\lambda$, $\forall \lambda \in [1,2]$.

Then for almost all $\lambda \in [1,2]$*, there exists a sequence* $\{u_n\}$ *such that*

$$\sup_n \|u_n\| < \infty, \qquad \Phi_\lambda'(u_n) \to 0, \qquad \Phi_\lambda(u_n) \to C_\lambda,$$

where

$$C_\lambda := \inf_{h \in \Gamma} \sup_{u \in Q} \Phi_\lambda\big(h(1,u)\big) \in \Big[\inf_B \Phi_\lambda, \sup_{\bar{Q}} \Phi\Big].$$

3. Critical points of Φ_n

In order to apply Lemma 2.1, we consider

$$\Phi_{n,\lambda}(u) := \frac{1}{2}\|u^+\|_X^2 - \lambda\left(\frac{1}{2}\|u^-\|_X^2 + \Psi_n(u)\right).$$

It is easy to see that $\Phi_{n,\lambda}$ verifies conditions (a), (b) and (c) in Lemma 2.1, since the norm $\|\cdot\|_X$ is equivalent to the H^1-norm on X_n. We still need to verify (d). Indeed, we have

Lemma 3.1. *Under assumptions (V_1) and (G_1)–(G_4), there hold:*

(i) *There exists $\rho > 0$ independent of $\lambda \in [1,2]$ such that $\kappa := \inf \Phi_\lambda(S_\rho X^+) > 0$ where $S_\rho X^+ := \{z \in X^+\colon \|z\|_X = \rho\}$.*
(ii) *For fixed $z_0 \in X^+$ with $\|z_0\|_X = 1$ and any $\lambda \in [1,2]$, there is $R_n > \rho > 0$ such that $\sup \Phi_{n,\lambda}(\partial Q_n) \leqslant 0$ where $Q_n = \{z = v + sz_0\colon v \in X_n^-,\ s \geqslant 0,\ \|z\|_X < R_n\}$.*

Proof. (i) Under assumptions (G_2), (G_3), we know for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|G(x,u)| \leqslant \varepsilon|u|^2 + C_\varepsilon|u|^p$. Hence, for any $u \in X^+$,

$$\Phi_{n,\lambda}(u) \geqslant \frac{1}{2}\|u\|_X^2 - \lambda\varepsilon\|u\|_X^2 - C'_\varepsilon\|u\|_X^p,$$

which implies the conclusion.

(ii) The proof for the case $\lambda = 1$ is contained in [16], we outline here for the completeness of the paper.

Suppose by contradiction that there exists $u_m \in X_n^- \oplus \mathbb{R}^+ z_0$ such that $\Phi_{n,\lambda}(u_m) > 0$ for all n and $\|u_m\|_X \to \infty$ as $m \to \infty$. Set $w_m = u_m/\|u_m\|_X = s_m z_0 + w_m^-$, then $1 = \|w_m\|_X = s_m + \|w_m^-\|_X$ and

$$0 < \frac{\Phi_{n,\lambda}(u_m)}{\|u_m\|_X^2} = \frac{1}{2}\left(s_m^2 - \lambda\|w_m^-\|_X^2\right) - \lambda\int\limits_{\mathbb{R}^N}\frac{G(x,u_m)}{u_m^2}w_m^2\,dx. \tag{3.1}$$

From (G_4), we know $G(x,u) \geqslant 0$, therefore

$$\|w_m^-\|_X^2 \leqslant \lambda\|w_m^-\|_X^2 < s_m^2 = 1 - \|w_m^-\|_X^2,$$

consequently we know $\|w_m^-\|_X \leqslant \frac{1}{\sqrt{2}}$ and $\frac{1}{1-\sqrt{2}} \leqslant s_m \leqslant 1$ for all m. Going to a subsequence if necessary, we may assume $s_m \to s > 0$, $w_m \rightharpoonup w$ and $w_m^-(x) \to w^-(x)$ a.e. in $\mathbb{R}^N$. Hence $w = sz_0 + w^-(x) \neq 0$ and therefore $|u_m| = \|u_m\|_X|w_m| \to +\infty$, as $m \to +\infty$. By $G(x,u) \geqslant 0$ again and Fatou's lemma, we obtain

$$\int\limits_{\mathbb{R}^N}\frac{G(x,u_m)}{u_m^2}w_m^2 \to +\infty \quad \text{as } m \to +\infty,$$

this contradicts to (3.1). □

Remark 3.2. By the proof of Lemma 3.1(ii), it is easy to see that there is an R independent of n such that conclusion (ii) still holds.

Applying Lemma 2.1, we readily obtain the following facts.

Lemma 3.3. *Under assumptions (V_1) and (G_1)–(G_4), for almost all $\lambda \in [1,2]$, there exists a sequence $\{u_m\}$ such that*

$$\sup_m\|u_m\|_X < \infty, \qquad \Phi'_{n,\lambda}(u_m) \to 0, \qquad \Phi_{n,\lambda}(u_m) \to C_\lambda \in \left[\kappa, \sup_{\bar{Q}_n}\Phi_n\right].$$

Lemma 3.4. *Under assumptions (V_1) and (G_1)–(G_4), for almost all $\lambda \in [1,2]$, there exists a u_λ such that*

$$\Phi'_{n,\lambda}(u_\lambda) = 0, \qquad \Phi_{n,\lambda}(u_\lambda) \leqslant \sup_{\bar{Q}_n}\Phi_n.$$

Proof. Let $\{u_m\}$ be the sequence obtained in Lemma 3.3, write $u_m = u_m^+ + u_m^-$ with $u_m^\pm \in X_n^\pm$. Since $\{u_m\}$ is bounded, we have either $\{u_m^+\}$ is vanishing, i.e.,

$$\lim_{m\to\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{B_1(y)}\left(u_m^+\right)^2 = 0$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $y_m \in \mathbb{Z}^N$ such that

$$\lim_{m\to\infty} \int\limits_{B_r(y_m)} \left(u_m^+\right)^2 \geqslant \delta.$$

If $\{u_m^+\}$ is vanishing, by Lion's concentration compactness principle [11], we have that $u_m^+ \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (2, 2^*)$. Since for any $\varepsilon > 0$ there exists C_ε such that $|g(x,u)| \leqslant \varepsilon|u| + C_\varepsilon |u|^{p-1}$, by Hölder's inequality, we know

$$\int\limits_{\mathbb{R}^N} \left|g(x,u_m)u_m^+\right| \leqslant \varepsilon \int\limits_{\mathbb{R}^N} |u_m|\left|u_m^+\right| + C_\varepsilon \int\limits_{\mathbb{R}^N} |u_m|^{p-1}\left|u_m^+\right| \to 0.$$

Therefore, we have

$$\Phi_{n,\lambda}(u_m) \leqslant \frac{1}{2}\left\|u_m^+\right\|_X^2 = \Phi'_{n,\lambda}(u_m)u_m^+ + \lambda \int\limits_{\mathbb{R}^N} g(x,u_m)u_m^+ \to 0,$$

this contradicts with the fact that $\Phi_{n,\lambda}(u_m) \geqslant \kappa$. Hence $\{u_m^+\}$ must be non-vanishing. Let us define $v_m = u_m(\cdot - y_m)$, then

$$\int\limits_{B_r(0)} \left(v_m^+\right)^2 \geqslant \frac{\delta}{2},$$

since $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ are both invariant under translation, we know

$$\Phi'_{n,\lambda}(v_m) \to 0 \quad \text{and} \quad \Phi_{n,\lambda}(v_m) \to C_\lambda.$$

Since v_m is still bounded, we may assume $v_m^+ \rightharpoonup u_\lambda^+$, $v_m^- \rightharpoonup u_\lambda^-$. From $v_m^+ \to u_\lambda^+$ in $L^2_{loc}(\mathbb{R}^N)$, we know $u_\lambda^+ \neq 0$. Moreover

$$\left(\Phi'_{n,\lambda}(u_\lambda), \varphi\right) = \lim_{m\to\infty}\left(\Phi'_{n,\lambda}(v_m), \varphi\right) = 0, \quad \forall \varphi \in X_n,$$

i.e. u_λ is a critical point of $\Phi_{n,\lambda}$.

From assumption (G_4), it is easy to see

$$\frac{1}{2}g(x,u)u - G(x,u) \geqslant 0, \quad \text{for all } u \in X_n.$$

Thus applying Fatou's lemma, we have

$$\begin{aligned}
\sup_{\bar{Q}_n} \Phi_{n,\lambda} \geqslant C_{n,\lambda} &= \lim_{m\to\infty}\left(\Phi_{n,\lambda}(u_m) - \frac{1}{2}\left(\Phi'_{n,\lambda}(u_m), u_m\right)\right) \\
&= \lim_{m\to\infty} \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}g(x,v_m)v_m - G(x,v_m)\right) \\
&\geqslant \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}g(x,u_\lambda)u_\lambda - G(x,u_\lambda)\right) = \Phi_{n,\lambda}(u_\lambda),
\end{aligned}$$

the proof is completed. □

We need the following two lemmas.

Lemma 3.5. *(See [16].) Let $u, s, v \in \mathbb{R}$ be numbers with $s \geqslant -1$ and $w := su + v \neq 0$ and let $x \in \mathbb{R}^N$. Then*

$$g(x,u)\left[s\left(\frac{s}{2}+1\right)u + (1+s)v\right] + G(x,u) - G(x,u+w) < 0.$$

The following proposition for $\{u_\lambda\}$ obtained in Lemma 3.4 is proved for $\lambda = 1$ in [16].

Lemma 3.6. *Let $\{u_\lambda\}$ be the critical point of $\Phi_{n,\lambda}$ obtained in Lemma* 3.4*, we have*

$$\Phi_{n,\lambda}(u_\lambda + w) < \Phi_{n,\lambda}(u_\lambda) \quad \textit{for any } w \in \Sigma_n := \left\{su_\lambda + v\colon\ s \geqslant -1,\ v \in X_n^-\right\},\ w \neq 0.$$

Proof. We rewrite $\Phi_{n,\lambda}$ by

$$\Phi_{n,\lambda}(u)=\frac{1}{2}\big(Au^+,u^+\big)+\frac{\lambda}{2}\big(Au^-,u^-\big)-\lambda\int_{\mathbb{R}^N}G(x,u).$$

Since $\Phi'_{n,\lambda}(u_\lambda)=0$, we have

$$\begin{aligned}0&=\left(\Phi'_\lambda(u_\lambda),\frac{2s+s^2}{2}u_\lambda+(1+s)v\right)\\&=\frac{2s+s^2}{2}\big(Au_\lambda^+,u_\lambda^+\big)+\lambda\frac{2s+s^2}{2}\big(Au_\lambda^-,u_\lambda^-\big)+\lambda(1+s)\big(Au_\lambda^-,v\big)-\lambda\int_{\mathbb{R}^N}g(x,u_\lambda)\left(\frac{2s+s^2}{2}u_\lambda+(1+s)v\right).\end{aligned}\tag{3.2}$$

From Lemma 3.5 and (3.2), we know

$$\begin{aligned}\Phi_{n,\lambda}(u_\lambda+w)-\Phi_{n,\lambda}(u_\lambda)&=\frac{1}{2}\big\{\big(\big(A(1+s)u_\lambda^+,(1+s)u_\lambda^+\big)\big)-\big(Au_\lambda^+,u_\lambda^+\big)\big\}\\&\quad+\frac{\lambda}{2}\big\{\big(A\big((1+s)u_\lambda^-+v\big),(1+s)u_\lambda^-+v\big)-\big(Au_\lambda^-,u_\lambda^-\big)\big\}\\&\quad+\lambda\left\{\int_{\mathbb{R}^N}G(x,u_\lambda)-\int_{\mathbb{R}^N}G(x,u_\lambda+w)\right\}\\&=\frac{2s+s^2}{2}\big(Au_\lambda^+,u_\lambda^+\big)+\lambda\frac{2s+s^2}{2}\big(Au_\lambda^-,u_\lambda^-\big)+\frac{\lambda}{2}(Av,v)+\lambda(1+s)\big(Au_\lambda^-,v\big)\\&\quad+\lambda\left\{\int_{\mathbb{R}^N}G(x,u_\lambda)-\int_{\mathbb{R}^N}G(x,u_\lambda+w)\right\}\\&=\frac{\lambda}{2}(Av,v)+\lambda\int_{\mathbb{R}^N}\left(g(x,u_\lambda)\left[s\left(\frac{s}{2}+1\right)u_\lambda+(1+s)v\right]+G(x,u_\lambda)-G(x,u_\lambda+w)\right)\\&<0.\qquad\square\end{aligned}$$

Lemma 3.7. *Under assumptions (V_1) and (G_1)–(G_4), there exist $\lambda_m\to 1$ and a sequence $\{u_{\lambda_m}\}$ such that*

$$\Phi'_{n,\lambda_m}(u_{\lambda_m})=0,\qquad \Phi_{n,\lambda_m}(u_{\lambda_m})\leqslant\sup_{\bar{Q}_n}\Phi_n.\tag{3.3}$$

Moreover $\{u_{\lambda_m}\}$ is bounded.

Proof. The existence of $\{u_{\lambda_m}\}$ such that

$$\Phi'_{n,\lambda_m}(u_{\lambda_m})=0,\qquad \Phi_{n,\lambda_m}(u_{\lambda_m})\leqslant\sup_{\bar{Q}_n}\Phi_n$$

is the direct consequence of Lemma 3.4. To prove the boundedness of $\{u_{\lambda_m}\}$, arguing by contradiction, suppose that $\|u_{\lambda_m}\|_X\to\infty$. Since $\Phi_{n,\lambda_m}(u_{\lambda_m})\geqslant 0$, we know $\|u_{\lambda_m}^+\|_X\geqslant\|u_{\lambda_m}^-\|_X$. Let $v_{\lambda_m}:=u_{\lambda_m}/\|u_{\lambda_m}\|_X$. Then $\|v_{\lambda_m}^+\|_X^2\geqslant\frac{1}{2}$ and $v_{\lambda_m}\rightharpoonup v_n\in X_n$, $v_{\lambda_m}(x)\to v_n(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. We have either $\{v_{\lambda_m}^+\}$ is vanishing, i.e.,

$$\lim_{m\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}\big|v_{\lambda_m}^+\big|^2=0$$

or non-vanishing, i.e., there exist $r,\delta>0$ and a sequence $y_m\in\mathbb{Z}^N$ such that

$$\lim_{m\to\infty}\int_{B_r(y_m)}\big|v_{\lambda_m}^+\big|^2\geqslant\delta.$$

If $v_{\lambda_m}^+$ is vanishing, Lion's concentration compactness principle implies $v_{\lambda_m}^+\to 0$ in $L^r(R^N)$ for $r\in(2,2^*)$. Therefore assumption (G_1) and Lebesgue Dominated Convergence Theorem imply that $\int_{\mathbb{R}^N}G(x,Tv_{\lambda_m}^+)\to 0$ for any $T\in\mathbb{R}$. From Lemma 3.6, we know that

$$\begin{aligned}\sup_{\bar{Q}_n}\Phi_n &\geqslant \Phi_{n,\lambda_m}(u_{\lambda_m}) \geqslant \Phi_{\lambda_n}\bigl(Tv_{\lambda_m}^+\bigr)\\ &= \frac{T^2}{2}\bigl\|v_{\lambda_m}^+\bigr\|_X^2 - \lambda_m\int_{\mathbb{R}^N} G\bigl(x, Tv_{\lambda_m}^+\bigr)\\ &\geqslant \frac{T^2}{4} - 2\int_{\mathbb{R}^N} G\bigl(x, Tv_{\lambda_m}^+\bigr) \to \frac{T^2}{4},\end{aligned}$$

we arrive a contradiction if T is large enough. Hence non-vanishing must hold and the invariance of Φ_{n,λ_m} under translation implies y_m can be selected to be bounded. Then $v_{\lambda_m}^+ \to v_n^+$ in $L^2_{loc}(\mathbb{R}^N)$ with $v_n^+ \neq 0$ and $|u_{\lambda_m}(x)| \to \infty$, as $m \to \infty$. It follows again from (G_3) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{G(x, u_{\lambda_m})}{u_{\lambda_m}^2} v_{\lambda_m}^2 \to +\infty \quad \text{as } m \to +\infty,$$

therefore

$$0 \leqslant \frac{\Phi_{n,\lambda_m}(u_{\lambda_m})}{\|u_{\lambda_m}\|_X^2} = \frac{1}{2}\bigl\|v_{\lambda_m}^+\bigr\|_X^2 - \lambda\left(\frac{1}{2}\bigl\|v_{\lambda_m}^-\bigr\|_X^2 + \int_{\mathbb{R}^N} \frac{G(x, u_{\lambda_m})}{u_{\lambda_m}^2} v_{\lambda_m}^2\right) \to -\infty,$$

as $m \to \infty$, which is a contradiction. Thus we have the conclusion. □

Corollary 3.8. *If $\{u_{\lambda_m}\}$ is the sequence obtained in Lemma* 3.7*, then it is also a $(P.S)$ sequence for Φ_n satisfying*

$$\lim_{m\to\infty}\Phi_n'(u_{\lambda_m}) = 0, \qquad \lim_{m\to\infty}\Phi_n(u_{\lambda_m}) \leqslant \sup_{\bar{Q}_n}\Phi_n. \tag{3.4}$$

Proof. Since $\{u_{\lambda_m}\}$ is bounded, from (3.3), we know

$$\lim_{m\to\infty}\Phi_n(u_{\lambda_m}) = \lim_{m\to\infty}\left(\Phi_{n,\lambda_m}(u_{\lambda_m}) + (\lambda_m - 1)\left(\frac{1}{2}\bigl\|u_{\lambda_m}^-\bigr\|_X^2 + \int_{\mathbb{R}^N} G(x, u_{\lambda_m})\right)\right)$$

and noting that the convergence is uniform in $\|\varphi\| \leqslant 1$,

$$\lim_{m\to\infty}\bigl(\Phi_n'(u_{\lambda_m}), \varphi\bigr) = \lim_{m\to\infty}\left(\bigl(\Phi_{n,\lambda_m}'(u_{\lambda_m}), \varphi\bigr) + (\lambda_m - 1)\left(\bigl(u_{\lambda_m}^-, \varphi^-\bigr) + \int_{\mathbb{R}^N} g(x, u_{\lambda_m})\varphi\right)\right),$$

we obtain the conclusion. □

Lemma 3.9. *For every $n \in N$ there is a nontrivial critical point v_n in X_n for Φ_n.*

Proof. From (3.4), we know u_{λ_m} is bounded $(P.S)$ sequence for Φ_n. We have either $\{u_{\lambda_m}\}$ is vanishing, i.e.,

$$\lim_{m\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} (u_{\lambda_m})^2 = 0$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $y_m \in \mathbb{Z}^N$ such that

$$\lim_{m\to\infty}\int_{B_r(y_m)} (u_{\lambda_m})^2 \geqslant \delta.$$

If u_{λ_m} is vanishing, then by Lion's concentration compactness principle again, we have that $u_{\lambda_m} \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. However, for any $\varepsilon > 0$ there exists C_ε such that $|g(x,u)| \leqslant \varepsilon|u| + C_\varepsilon|u|^{p-1}$, from the fact that $(\Phi_{n,\lambda_m}'(u_{\lambda_m}), u_{\lambda_m}^+) = 0$ and Hölder's inequality, we know

$$\begin{aligned}\|u_{\lambda_m}^+\|_X^2 &= \lambda \int_{\mathbb{R}^N} g(x, u_{\lambda_m}) u_{\lambda_m}^+ \\ &\leqslant \varepsilon \int_{\mathbb{R}^N} |u_{\lambda_m}| |u_{\lambda_m}^+| + C_\varepsilon \int_{\mathbb{R}^N} |u_{\lambda_m}|^{p-1} |u_{\lambda_m}^+| \\ &\leqslant \varepsilon \|u_{\lambda_m}\|_X \|u_{\lambda_m}^+\|_X + C'_\varepsilon \|u_{\lambda_m}\|_X^{p-1} \|u_{\lambda_m}^+\|_X. \end{aligned} \tag{3.5}$$

Similarly, we have

$$\|u_{\lambda_m}^-\|_X^2 \leqslant \varepsilon \|u_{\lambda_m}\|_X \|u_{\lambda_m}^-\|_X + C'_\varepsilon \|u_{\lambda_m}\|_X^{p-1} \|u_{\lambda_m}^-\|_X. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|u_{\lambda_m}\|_X^2 \leqslant \varepsilon \|u_{\lambda_m}\|_X^2 + C'_\varepsilon \|u_{\lambda_m}\|_X^p$$

which means $\|u_{\lambda_m}\|_X \geqslant c$ for some constant c, hence the vanishing case does not hold. Let us now define $v_{\lambda_m} = u_{\lambda_m}(\cdot - y_m)$, then

$$\int_{B_r(0)} (v_{\lambda_m})^2 \geqslant \frac{\delta}{2}.$$

Since v_{λ_m} is also bounded, we may assume $v_{\lambda_m} \rightharpoonup v_n$ with $v_{\lambda_m} \to v_n$ in $L^2_{loc}(\mathbb{R}^N)$ and $v_n \neq 0$. Φ_n and Φ'_n being both invariant by translation, we know

$$\Phi'_n(v_{\lambda_m}) \to 0, \quad \text{as } m \to \infty,$$

consequently $\Phi'_n(v_n) = 0$ and $\Phi_n(v_n) \leqslant \sup_{\bar{Q}_n} \Phi_n$. □

4. Proof of the main result

Lemma 4.1. *For every $n \in N$, let v_n be the critical point obtained in Lemma* 3.9 *for Φ_n, then $\{\Phi_n(v_n)\}$ is bounded.*

Proof. From Remark 3.2, it is only left to notice that

$$\Phi_n(v_n) \leqslant \sup_{\bar{Q}_n} \Phi_n \leqslant \frac{1}{2} R^2. \qquad \square$$

Lemma 4.2. *For every $n \in N$, let v_n be the critical point obtained in Lemma* 3.9 *for Φ_n then $\{v_n\}$ is bounded in X_μ.*

Proof. Since $\Phi'_n(v_n) = 0$ and $\kappa \leqslant \Phi_n(v_n) \leqslant \frac{R^2}{2}$, we know there exist a subsequence still denoted by v_n and $c \in [\kappa, \frac{R^2}{2}]$ such that $\Phi_n(v_n) \to c$, hence $\{v_n\}$ is a $(P.S)^*_c$ sequence for Φ. By $\|v_n^+\|_X = \|v_n^-\|_X + \int_{\mathbb{R}^N} g(x, v_n) v_n \geqslant C \int_{\mathbb{R}^N} |v_n|^\mu$, we only need to show v_n is bounded in X. Suppose v_n is unbounded in X, thus $\|v_n\|_X \to +\infty$. Since $\Phi(v_n) \geqslant 0$, we know $\|v_n^+\|_X \geqslant \|v_n^-\|_X$. Let $w_n := v_n / \|v_n\|_X$. Then $\|w_n^+\|_X^2 \geqslant \frac{1}{2}$ and $w_n \rightharpoonup w$, $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. We have either w_n^+ is vanishing or non-vanishing. If w_n^+ is vanishing, Lion's concentration compactness principle implies $w_n^+ \to 0$ in $L^r(R^N)$ for $r \in (2, 2^*)$. Therefore assumption (G_1) and Lebesgue Dominated Convergence Theorem imply that $\int_{\mathbb{R}^N} G(x, T w_n^+) \to 0$ for any $T \in \mathbb{R}$. From Lemmas 3.6 and 4.2, we know

$$\begin{aligned}\frac{R^2}{2} &\geqslant \Phi(v_n) = \Phi_n(v_n) \geqslant \Phi_n(T w_n^+) \\ &= \frac{T^2}{2} \|w_n^+\|_X^2 - \int_{\mathbb{R}^N} G(x, T w_n^+) \\ &\geqslant \frac{T^2}{4} - \int_{\mathbb{R}^N} G(x, T w_n^+) \to \frac{T^2}{4}, \end{aligned}$$

we arrive a contradiction if T is large enough. The non-vanishing case is same to Lemma 3.7. Thus we know that $\{v_n\}$ is bounded in X_μ. □

Proof of Theorem 1.1. Since v_n is bounded in X_μ, as in Lemma 3.9, a concentration compactness argument shows $v_n \rightharpoonup v$ with $v^+ \neq 0$. Letting $\varphi \in C_0^\infty(\mathbb{R}^N)$, by Lemma 2.1 in [4], we have

$$\left| \int_{\mathbb{R}^N} g(x, v_n)(Id - Q_n)\varphi \right| \leqslant \varepsilon \int_{\mathbb{R}^N} |v_n| \left|(Id - Q_n)\varphi\right| + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{p-1} \left|(Id - Q_n)\varphi\right| \to 0, \quad \text{as } n \to \infty.$$

Now

$$\begin{aligned}(Av_n, \varphi) &= (Av_n, Q_n\varphi) \\ &= \left(\Phi_n'(v_n), Q_n\varphi\right) + \int_{\mathbb{R}^N} g(x, v_n)\varphi - \int_{\mathbb{R}^N} g(x, v_n)(Id - Q_n)\varphi\end{aligned}$$

and therefore, taking $n \to \infty$, we have

$$(Av, \varphi) = \int_{\mathbb{R}^N} g(x, v)\varphi.$$

This implies v is a weak solution of problem (1.1) and $v \neq 0$ in X_μ. $\square$

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